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L-functions for images of graph coverings by some operations ☆

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Abstract

We express the L-functions of the line graph and the middle graph of a regular covering of a regular graph G by using some characteristic polynomials. As a corollary, we express the zeta functions of the line graph and the middle graph of a regular covering of G by using the characteristic polynomial of that regular covering.

Keywords: L-function; Graph covering; Line graph

1. Introduction

Graphs and digraphs treated here are finite and simple. Let G be a connected graph and D the symmetric digraph corresponding to G. A *path P of length n* in D (or G) is a sequence $P=(v_0,v_1,\ldots,v_{n-1},v_n)$ of $n+1$ vertices and n arcs (or edges) such that consecutive vertices share an arc (edge) (we do not require that all vertices are distinct). Also, P is called a *(v_0,v_n)-path*. We say that a path has a *backtracking* if a subsequence of the form $\ldots,x,y,x,\ldots$ appears. A (v,w)-path is called a *v-cycle* (or *v-closed path*) if $v=w$. The *inverse cycle* of a cycle $C=(v,v_1,\ldots,v_{n-1},v)$ is the cycle $C^{-1}=(v,v_{n-1},\ldots,v_1,v)$.

We introduce an equivalence relation between cycles. Such two cycles $C_1=(v_1,\ldots,v_m)$ and $C_2=(w_1,\ldots,w_m)$ are called *equivalent* if $w_j=v_{j+k}$ for all j. The

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inverse cycle of C is not equivalent to C. Let $[C]$ be the equivalence class which contains a cycle C. Let B^r be the cycle obtained by going r times around a cycle B. Such a cycle is called a *multiple* of B. A cycle C is *primitive* if both C and C^2 have no backtracking, and it is not a multiple of a strictly smaller cycle. Note that each equivalence class of primitive cycles of a graph G corresponds to a unique congugacy class of the fundamental group $\pi_1(G,v)$ of G for a vertex v of G.

Mizuno and Sato [12] introduced an L-function of G. Let G be a graph and A a finite group. Let $D(G)$ be the arc set of the symmetric digraph corresponding to G. Then a mapping $\alpha : D(G) \to A$ is called an *ordinary voltage assignment* if $\alpha(v,u) = \alpha(u,v)^{-1}$ for each $(u,v) \in D(G)$. Moreover, we define the *net voltage* $\alpha(P)$ of each path $P = (v_1,\dots,v_l)$ of G by $\alpha(P) = \alpha(v_1,v_2),\dots,\alpha(v_{l-1},v_l)$ (see [5]). Furthermore, let ρ be an irreducible representation of A. The *L-function* of G associated to ρ and α is defined to be the function of $u \in \mathbf{C}$ with u sufficiently small, given by

$$\mathbf{Z}(u,\rho,\alpha) = \mathbf{Z}_G(u,\rho,\alpha) = \prod_{|C|} \det(\mathbf{I}_f - \rho(\alpha(C))u^{|C|})^{-1},$$

where $f = \deg \rho$ and $[C]$ runs over all equivalence classes of primitive cycles of G (c.f. [8,9,15]).

Let $\rho = 1$ be the identity representation of A. Then, note that the L-function of G associated to 1 and α is the (Ihara) zeta function $\mathbf{Z}(G,u)$ of a graph G (see [9]):

$$\mathbf{Z}_G(u,1,\alpha) = \mathbf{Z}(G,u) = \prod_{|C|} (1 - u^{|C|})^{-1}.$$

Let G be a connected graph with n vertices $v_1,\dots,v_n$. The *adjacency matrix* $\mathbf{A} = \mathbf{A}(G) = (a_{ij})$ is the square matrix such that $a_{ij} = 1$ if v_i and v_j are adjacent, and $a_{ij} = 0$ otherwise. Let $\mathbf{D} = (d_{ij})$ be the diagonal matrix with $d_{ii} = \deg_G v_i$, and $\mathbf{Q} = \mathbf{D} - \mathbf{I}$. For a positive integer m, set $\mathbf{Q}_m = \mathbf{I}_m \otimes \mathbf{Q}$, where $\mathbf{I}_m$ is the indentity matrix of order m. The *Kronecker product* $\mathbf{A} \otimes \mathbf{B}$ of matrices $\mathbf{A}$ and $\mathbf{B}$ is considered as the matrix $\mathbf{A}$ having the element a_{ij} replaced by the matrix $a_{ij}\mathbf{B}$.

Mizuno and Sato [12] showed that the reciprocal of any L-function of G is an explicit polynomial.

Theorem 1 (Mizuno and Sato [12]). *Let G be a connected graph with n vertices and l edges, A a finite group and $\alpha : D(G) \to A$ an ordinary voltage assignment. Furthermore, let ρ be an irreducible representation of A, and f the degree of ρ. For $g \in A$, the matrix $\mathbf{A}_g = (a_{uv}^{(g)})$ is defined as follows*:

$$a_{uv}^{(g)} := \begin{cases} 1 & \text{if } \alpha(u,v) = g \text{ and } (u,v) \in D(G), \\ 0 & \text{otherwise.} \end{cases}$$

Then the reciprocal of the L-function of G associated to ρ and α is

$$\mathbf{Z}_G(u,\rho,\alpha)^{-1} = (1-u^2)^{(l-n)f} \det\left(\mathbf{I}_{fn} - u\sum_{g\in A} \rho(g) \otimes \mathbf{A}_g + u^2\mathbf{Q}_f\right).$$

Typical operations of graphs are the line graph $L(G)$ and the middle graph $M(G)$ of a graph G (see [3]).

Let G be a graph. Then the *line graph* $L(G)$ is the graph whose vertex set is the edge set $E(G)$ of G, with two vertices of $L(G)$ being adjacent if and only if the corresponding edges in G have a vertex in common. The *middle graph* $M(G)$ is the graph obtained from G inserting a new vertex into every edge of G and by joining by edges those pairs of these new vertices which lie on adjacent edges of G. The adjacency matricies $\mathbf{A}_L = \mathbf{A}(L(G))$ and $\mathbf{A}_M = \mathbf{A}(M(G))$ are given as follows:

$$\mathbf{A}_L = \mathbf{B}{}^{\mathrm{t}}\mathbf{B} - 2\mathbf{I}_l, \qquad \mathbf{A}_M = \begin{bmatrix} \mathbf{A}_L & \mathbf{B} \\ {}^{\mathrm{t}}\mathbf{B} & 0 \end{bmatrix}, \tag{1}$$

where $l = |E(G)|$ and $\mathbf{B} = (b_{ij})$ is the incidence matrix of G: $b_{ij} = 1$ if the edge e_i and the vertex v_j are incident, and $b_{ij} = 0$ otherwise. Furthermore, we have $\mathbf{A}(G) = {}^{\mathrm{t}}\mathbf{B}\mathbf{B} - \mathbf{D}$.

In this paper, we express the L-functions of the line graph and the middle graph of a regular graph G in terms of characteristic polynomials.

For a general theory of the representation of groups and graph coverings, the reader is referred to [2] and [5], respectively.

2. Zeta functions of regular coverings

Let G be a connected graph, and let $N(v) = \{w \in V(G) \mid vw \in E(G)\}$ for any vertex v in G. A graph H is called a *covering* of G with projection $\pi : H \to G$ if there is a surjection $\pi : V(H) \to V(G)$ such that $\pi|_{N(v')} : N(v') \to N(v)$ is a bijection for all vertices $v \in V(G)$ and $v' \in \pi^{-1}(v)$. When a finite group Π acts on a graph G, the *quotient graph* G/Π is a simple graph whose vertices are the Π-orbits on $V(G)$, with two vertices adjacent in G/Π if and only if some two of their representatives are adjacent in G. A covering $\pi : H \to G$ is said to be a *regular covering* of G if there is a subgroup B of the automorphism group $\mathrm{Aut}\, H$ of H acting freely on H without fixed points such that the quotient graph H/B is isomorphic to G.

Let G be a connected graph, let A be a finite group, and let $\alpha : D(G) \to A$ be an ordinary voltage assignment. The pair (G, α) is called an *ordinary voltage graph*. The *derived graph* G^α of the ordinary voltage graph (G, α) is defined as follows:

$$V(G^\alpha) = V(G) \times A \text{ and } ((u,h),(v,k)) \in D(G^\alpha)$$

$$\text{if and only if } (u,v) \in D(G) \text{ and } k = h\alpha(u,v),$$

where $V(G)$ is the vertex set of G. The *natural projection* $\pi : G^\alpha \to G$ is defined by $\pi(v,h) = v$ for all $(v,h) \in V(G) \times A$. The graph G^α is called an A-*covering* of G. The A-covering G^α is an $|A|$-fold regular covering of G. Every regular covering of G is an A-covering of G for some group A (see [4]).

Ihara [9] defined zeta functions of graphs, and showed that the reciprocals of zeta functions of regular graphs are explicit polynomials. Hashimoto [7] treated multivariable zeta functions of bipartite graphs. Bass [1] generalized Ihara's result on the zeta function of a regular graph to an irregular graph G, and showed that the reciprocal of the zeta function of G is given by

$$\mathbf{Z}(G,u)^{-1} = (1-u^2)^{r-1} \det(\mathbf{I} - u\mathbf{A} + u^2\mathbf{Q}),$$

where r is the Betti number of G. Stark and Terras [14] gave an elementary proof of this formula, and discussed three different zeta functions of any graph.

Furthermore, Mizuno and Sato [12] showed that the zeta function of a regular covering of G is a product of L-functions of G.

Theorem 2 (Mizuno and Sato [12]). *Let G be a connected graph, A a finite group and $\alpha : D(G) \to A$ an ordinary voltage assignment. Suppose that G^α is connected. Then we have*

$$\mathbf{Z}(G^\alpha, u) = \prod_\rho \mathbf{Z}_G(u, \rho, \alpha)^f,$$

where ρ runs over all irreducible representations of A, and $f = \deg \rho$.

Mizuno and Sato [11] expressed the characteristic polynomial of a regular covering of G by using that of G. Let $\Phi(\mathbf{F};\lambda) = \det(\lambda\mathbf{I} - \mathbf{F})$ for any.square matrix $\mathbf{F}$.

Theorem 3 (Mizuno and Sato [11]). *Let G be a connected graph, A a finite group and $\alpha : D(G) \to A$ an ordinary voltage assignment. Furthermore, let $\rho_1 = 1$, $\rho_2, \ldots, \rho_t$ be the irreducible representations of A, and f_i the degree of ρ_i for each i, where $f_1 = 1$. Then the characteristic polynomial of the A-covering G^α of G is*

$$\Phi(G^\alpha;\lambda) = \Phi(G;\lambda) \cdot \prod_{i=2}^{t} \Phi\left(\sum_{g\in A} \rho_i(g) \otimes \mathbf{A}_g;\lambda\right)^{f_i}.$$

3. *L*-functions of line graphs

Let G be a connected graph, A a finite group and $\alpha : D(G) \to A$ an ordinary voltage assignment. For $e = (u,v) \in D(G)$, let $\mathrm{o}(e) = u$ and $t(e) = v$. The inverse arc of e is denoted by e^{-1}.

The set $D(L(G))$ of arcs in the line graph $L(G)$ is given by

$$\{(e,f) \mid e \neq f^{-1}, t(e) = \mathrm{o}(f)\}.$$

Then we have $\mathrm{o}((e,f)) = [e]$ and $t((e,f)) = [f]$, where $[e]$ denotes the edge obtained from e by deleting its direction. Furthermore, $(e,f)^{-1} = (f^{-1}, e^{-1})$. Kotani and Sunada [10] showed that the line graph of the A-covering G^α of G is an A-covering of the line graph $L(G)$ of G.

Mizuno and Sato [13] determined a voltage assignment $\beta : D(L(G)) \to A$ such that $L(G^\alpha) = L(G)^\beta$.

Lemma 1. *Let G be a connected graph with n vertices $v_1,\dots,v_n$, A a finite group and $\alpha: D(G)\to A$ an ordinary voltage assignment. For each edge $v_iv_j \in E(G)$, let $e_{ij}=(v_i,v_j)$.* Furthermore, let *$\alpha_L: D(L(G))\to A$ be defined by*

$$\alpha_L([e_{ij}],[e_{jk}]) := \begin{cases} \alpha(e_{ij}) & \text{if } i<j<k, \\ \alpha(e_{ij})\alpha(e_{jk}) & \text{if } i<j \text{ and } j>k, \\ \alpha(e_{jk}) & \text{if } i>j>k, \\ 1 & \text{if } i>j \text{ and } j<k. \end{cases}$$

Then $L(G^{\alpha})=L(G)^{\alpha_L}$.

Let G be a connected graph with n vertices $v_1,\dots,v_n$ and l edges $e_1,\dots,e_l$. For $g\in A$, the matrix $(\mathbf{A}_L)_g=(a_{ef}^{(g)})$ is defined as follows:

$$a_{ef}^{(g)} := \begin{cases} 1 & \text{if } \alpha_L(e,f)=g \text{ and } (e,f)\in D(L(G)), \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, let $\mathbf{D}_L=(d_{ij})$ be the diagonal matrix with $d_{ii}=\deg_{L(G)} e_i$, and $\mathbf{Q}_L=\mathbf{D}_L-\mathbf{I}_l$.

Let ρ be an irreducible representation of A, and f the degree of ρ. Then, let $\mathbf{B}_\rho$ be the $lf\times nf$ matrix defined as follows:

$$(\mathbf{B}_\rho)_{ij} := \begin{cases} \mathbf{I}_f & \text{if } e_i=(v_j,v_k) \text{ and } j<k, \\ \rho(\alpha(e_{kj})) & \text{if } e_i=(v_k,v_j) \text{ and } j>k, \\ \mathbf{0}_f & \text{otherwise,} \end{cases}$$

where $(\mathbf{B}_\rho)_{ij}$ is the (i,j)-block of $\mathbf{B}_\rho$. Then we have

$$\mathbf{B}_\rho {}^{t}\bar{\mathbf{B}}_\rho = \sum_{g\in A} (\mathbf{A}_L)_g \otimes \rho(g) + 2\mathbf{I}_{lf} \tag{2}$$

and

$${}^{t}\bar{\mathbf{B}}_\rho \mathbf{B}_\rho = \sum_{g\in A} \mathbf{A}_g \otimes \rho(g) + \mathbf{D}\otimes \mathbf{I}_f, \tag{3}$$

where ${}^{t}\bar{\mathbf{B}}_\rho$ is the conjugate transpose of $\mathbf{B}_\rho$.

We express the L-function of the line graph $L(G)$ for a regular graph G in terms of characteristic polynomials.

Theorem 4. *Let G be a connected r-regular graph with n vertices and l edges, A a finite group and $\alpha: D(G)\to A$ an ordinary voltage assignment. Furthermore, let ρ be an irreducible representation of A, and f the degree of ρ. Suppose that the A-covering*

G^α of G is connected. Then

$$\mathbf{Z}_{L(G)}(u,\rho,\alpha_L)^{-1} = (1-u^2)^{(r-2)lf} u^{nf} (1+2u+(2r-3)u^2)^{(l-n)f}$$
$$\Phi\left(\sum_{g\in A} \rho(g)\otimes \mathbf{A}_g; \frac{1+(2-r)u+(2r-3)u^2}{u}\right).$$

Proof. At first, both $L(G)$ and $L(G^\alpha)$ are $(2r-2)$-regular graphs. By Theorem 1, we have

$$\mathbf{Z}_{L(G)}(u,\rho,\alpha_L)^{-1} = (1-u^2)^{(r-2)lf} \det\left(\mathbf{I}_{lf} - u\sum_{g\in A} \rho(g)\otimes(\mathbf{A}_L)_g + u^2 a\mathbf{I}_{lf}\right),$$

where $a = 2r-3$. But, we have

$$\det\left(\mathbf{I}_{lf} - u\sum_{g\in A}\rho(g)\otimes(\mathbf{A}_L)_g + u^2 a\mathbf{I}_{lf}\right)$$
$$= u^{lf}\det\left(\frac{1+au^2}{u}\mathbf{I}_{lf} - \sum_{g\in A}(\mathbf{A}_L)_g\otimes\rho(g)\right).$$

Now, let

$$\mathbf{X} = \begin{bmatrix} \lambda\mathbf{I}_{nf} & -{}^{\mathrm{t}}\bar{\mathbf{B}}_\rho \\ \mathbf{0} & \mathbf{I}_{lf} \end{bmatrix}, \qquad \mathbf{Y} = \begin{bmatrix} \mathbf{I}_{nf} & {}^{\mathrm{t}}\bar{\mathbf{B}}_\rho \\ \mathbf{B}_\rho & \lambda\mathbf{I}_{lf} \end{bmatrix}.$$

Then we have

$$\mathbf{XY} = \begin{bmatrix} \lambda\mathbf{I}_{nf} - {}^{\mathrm{t}}\bar{\mathbf{B}}_\rho\mathbf{B}_\rho & \mathbf{0} \\ \mathbf{B}_\rho & \lambda\mathbf{I}_{lf} \end{bmatrix}$$

and

$$\mathbf{YX} = \begin{bmatrix} \lambda\mathbf{I}_{nf} & \mathbf{0} \\ \lambda\mathbf{B}_\rho & \lambda\mathbf{I}_{lf} - \mathbf{B}_\rho{}^{\mathrm{t}}\bar{\mathbf{B}}_\rho \end{bmatrix}.$$

Since $\det XY = \det YX$,

$$\lambda^{lf}\det(\lambda\mathbf{I}_{nf} - {}^{\mathrm{t}}\bar{\mathbf{B}}_\rho\mathbf{B}_\rho) = \lambda^{nf}\det(\lambda\mathbf{I}_{lf} - \mathbf{B}_\rho{}^{\mathrm{t}}\bar{\mathbf{B}}_\rho),$$

i.e.,

$$\det(\lambda\mathbf{I}_{lf} - \mathbf{B}_\rho{}^{\mathrm{t}}\bar{\mathbf{B}}_\rho) = \lambda^{(l-n)f}\det(\lambda\mathbf{I}_{nf} - {}^{\mathrm{t}}\bar{\mathbf{B}}_\rho\mathbf{B}_\rho). \tag{4}$$

By (2) and (4), we have

$$\det\left(\frac{1+au^2}{u}\mathbf{I}_{lf}-\sum_{g\in A}(\mathbf{A}_L)_g\otimes\rho(g)\right)=\det\left(\frac{1+au^2}{u}\mathbf{I}_{lf}-\mathbf{B}_\rho{}^{\mathrm{t}}\bar{\mathbf{B}}_\rho+2\mathbf{I}_{lf}\right)$$

$$=\det\left(\frac{1+2u+au^2}{u}\mathbf{I}_{lf}-\mathbf{B}_\rho{}^{\mathrm{t}}\bar{\mathbf{B}}_\rho\right)$$

$$=\left(\frac{1+2u+au^2}{u}\right)^{(l-n)f}$$

$$\det\left(\frac{1+2u+au^2}{u}\mathbf{I}_{nf}-{}^{\mathrm{t}}\bar{\mathbf{B}}_\rho\mathbf{B}_\rho\right).$$

By (3), we have

$$\det\left(\frac{1+au^2}{u}\mathbf{I}_{lf}-\sum_{g\in A}(\mathbf{A}_L)_g\otimes\rho(g)\right)$$

$$=\left(\frac{1+2u+au^2}{u}\right)^{(l-n)f}$$

$$\det\left(\frac{1+2u+au^2}{u}\mathbf{I}_{nf}-\sum_{g\in A}\mathbf{A}_g\otimes\rho(g)-r\mathbf{I}_{nf}\right)$$

$$=\left(\frac{1+2u+au^2}{u}\right)^{(l-n)f}\det\left(\frac{1+(2-r)u+au^2}{u}\mathbf{I}_{nf}-\sum_{g\in A}\mathbf{A}_g\otimes\rho(g)\right)$$

$$=\left(\frac{1+2u+au^2}{u}\right)^{(l-n)f}\Phi\left(\sum_{g\in A}\mathbf{A}_g\otimes\rho(g);\frac{1+(2-r)u+au^2}{u}\right).$$

Therefore, it follows that

$$\mathbf{Z}_{L(G)}(u,\rho,\alpha_L)^{-1}=(1-u^2)^{(r-2)lf}u^{nf}(1+2u+au^2)^{(l-n)f}$$

$$\Phi\left(\sum_{g\in A}\rho(g)\otimes\mathbf{A}_g;\frac{1+(2-r)u+au^2}{u}\right).\qquad\square$$

Mizuno and Sato [13] expressed the zeta function of the line graph $L(G^\alpha)$ for a regular graph G by using the characteristic polynomial of G^α.

Corollary 1 (Mizuno and Sato [13]). *Let G be a connected r-regular graph with n vertices and l edges, A a finite group and $\alpha : D(G) \to A$ an ordinary voltage assignment. Set $|A| = m$. Suppose that the A-covering G^α of G is connected. Then*

$$\mathbf{Z}(L(G^\alpha), u)^{-1} = (1-u^2)^{(r-2)lm} u^{nm} (1+2u+(2r-3)u^2)^{(l-n)m} \Phi\left(G^\alpha; \frac{1+(2-r)u+(2r-3)u^2}{u}\right).$$

Proof. Let $\rho_1 = 1, \rho_2, \ldots, \rho_t$ be the irreducible representations of A, and f_i the degree of ρ_i for each i, where $f_1 = 1$, By Theorems 2, 4 and Lemma 1, we have

$$\begin{aligned}\mathbf{Z}(L(G^\alpha), u)^{-1} &= \mathbf{Z}(L(G)^{\alpha_L}, u)^{-1} = \prod_{i=1}^{t} \mathbf{Z}_{L(G)}(u, \rho_i, \alpha_L)^{-f_i} \\ &= \prod_{i=1}^{t} \left\{ (1-u^2)^{(r-2)lf_i} u^{nf_i} (1+2u+au^2)^{(l-n)f_i} \right. \\ &\quad \left. \Phi\left(\sum_{g\in A} \rho_i(g) \otimes \mathbf{A}_g; \frac{1+(2-r)u+au^2}{u}\right)\right\}^{f_i},\end{aligned}$$

where $a = 2r-3$. Since $1 + f_2^2 + \cdots + f_t^2 = m$,

$$\begin{aligned}\mathbf{Z}(L(G^\alpha), u)^{-1} &= (1-u^2)^{(r-2)lm} u^{nm} (1+2u+au^2)^{(l-n)m} \\ &\quad \Phi\left(G; \frac{1+(2-r)u+au^2}{u}\right) \prod_{i=2}^{t} \\ &\quad \Phi\left(\sum_{g\in A} \rho_i(g) \otimes \mathbf{A}_g; \frac{1+(2-r)u+au^2}{u}\right)^{f_i}\end{aligned}$$

By Theorem 3, the result follows. □

Now, we give an example. Let $G = K_3$ be the complete graph with three vertices v_1, v_2, v_3 and $A = Z_3 = \{1, \tau, \tau^2\} (\tau^3 = 1)$ the cyclic group of order 3. Furthermore, let $\alpha : D(K_3) \to Z_3$ be the ordinary voltage assignment such that $\alpha(v_1, v_2) = \tau$ and $\alpha(v_1, v_3) = \alpha(v_2, v_3) = 1$. Then, the A-covering K_3^α is the cycle graph C_9 with nine vertices. Furthermore, we have

$$\alpha_L([e_{12}], [e_{23}]) = \alpha(v_1, v_2) = \tau; \quad \alpha_L([e_{23}], [e_{31}]) = \alpha(v_2, v_3)\alpha(v_3, v_1) = 1;$$

$$\alpha_L([e_{31}], [e_{12}]) = 1.$$

Since $L(K_3^\alpha) = L(C_9) = C_9$ and $L(K_3) = K_3$, it is certain that $L(K_3^\alpha) = L(K_3)^{\alpha_L}$.

Now, we present the L-function $\mathbf{Z}_{L(K_3)}(u,\chi_1,\alpha_L)$ of $L(K_3)$ associated to χ_1 and α_L in three ways. The characters of $\mathbf{Z}_3$ are given as follows: $\chi_i(\tau^j)=(\zeta^i)^j$, $0\leqslant i,j\leqslant 2$, where $\zeta=(1+\sqrt{-3})/2$. All primitive cycles of $L(K_3)$ are C, C^{-1}, where $C=([e_{12}],[e_{23}],[e_{31}],[e_{12}])$. By the definition of the L-function of a graph,

$$\begin{aligned}Z_{L(K_3)}(u,\chi_1,\alpha_L)^{-1} &= (1-\chi_1(\alpha_L(C))u^3)(1-\chi_1(\alpha_L(C^{-1}))u^3)\\ &= (1-\zeta u^3)(1-\zeta^2 u^3)=1+u^3+u^6.\end{aligned}$$

Theorem 1 implies that

$$\begin{aligned}\mathbf{Z}_{L(K_3)}(u,\chi_1,\alpha_L)^{-1} &= \det\left(\mathbf{I}_3-u\sum_{i=0}^{2}\chi_1(\tau^i)(\mathbf{A}_L)_{\tau^i}+u^2\mathbf{I}_3\right)\\ &= \det\begin{bmatrix}1+u^2 & -\zeta u & -u\\ -\zeta^2 u & 1+u^2 & -u\\ -u & -u & 1+u^2\end{bmatrix}\\ &= 1+u^3+u^6.\end{aligned}$$

Next, by Theorem 3, we have

$$\Phi\left(\sum_{i=0}^{2}\chi_1(\tau^i)\mathbf{A}_{\tau^i};\lambda\right)=\det\begin{bmatrix}\lambda & -\zeta & -1\\ -\zeta^2 & \lambda & -1\\ -1 & -1 & \lambda\end{bmatrix}=\lambda^3-3\lambda+1.$$

By Theorem 4, it follows that

$$\begin{aligned}\mathbf{Z}_{L(K_3)}(u,\chi_1,\alpha_L)^{-1} &= u^3\Phi\left(\sum_{i=0}^{2}\chi_1(\tau^i)\mathbf{A}_{\tau^i};\frac{1+u^2}{u}\right)\\ &= (1+u^2)^2-3u^2(1+u^2)+u^3=1+u^3+u^6.\end{aligned}$$

4. *L*-functions of middle graphs

Let G be a connected graph with n vertices $v_1,\ldots,v_n$. Then the middle graph $M(G)$ of G is defined by $V(M(G))=V(G)\cup E(G)$; $E(M(G))=E(L(G))\cup\{ue \mid e\in E(G), v\in V(G)$ are incident in $G\}$. The *endline graph* G^+ of G is given as follows: $V(G^+)=\{v_1,\ldots,v_n,v_1',\ldots,v_n'\}$ and $E(G^+)=E(G)\cup\{v_1v_1',\ldots,v_nv_n'\}$. Hamada and Yoshimura [6] showed that $M(G)=L(G^+)$.

For a finite group A and an ordinary voltage assignment $\alpha: D(G)\to A$, Mizuno and Sato [13] showed that the middle graph of the A-covering G^α of G is an A-covering of the middle graph $M(G)$ of G.

Theorem 5 (Mizuno and Sato [13]). *Let G be a connected graph with n vertices $v_1,\dots,v_n$, A a finite group and $\alpha : D(G) \to A$ an ordinary voltage assignment. Let $v_1,\dots,v_n,v'_1,\dots,v'_n$ be an order in $V(G^+)$. Furthermore, let $\alpha_M : D(M(G)) \to A$ be defined by*

$$\alpha_M(u,v) := \begin{cases} \alpha_L(u,v) & \textit{if } (u,v) \in D(L(G)), \\ 1 & \textit{if } u = [e_{ij}],\ v = v_j v'_j \textit{ and } i > j, \\ a(e_{ij}) & \textit{if } u = [e_{ij}],\ v = v_j v'_j \textit{ and } i < j, \end{cases}$$

where $e_{ij} = (v_i, v_j)$. Then $M(G^\alpha) = M(G)^{\alpha_M}$.

For $g \in A$, the matrix $(\mathbf{A}_M)_g = (a^{(g)}_{uv})$ is defined as follows: $a^{(g)}_{uv} = 1$ if $\alpha_M(u,v) = g$ and $(u,v) \in D(M(G))$, and $a^{(g)}_{uv} = 0$ otherwise. Furthermore, let $\mathbf{D}_M = (d_{ij})$ be the diagonal matrix with $d_{ii} = \deg_{M(G)} e_i$ if $1 \leqslant i \leqslant l$, and $d_{ii} = \deg_{M(G)} v_{i-l}$ if $l+1 \leqslant i \leqslant l+n$, and $\mathbf{Q}_M = \mathbf{D}_M - \mathbf{I}_{l+n}$, where $V(G) = \{v_1,\dots,v_n\}$ and $E(G) = \{e_1,\dots,e_l\}$.

Now, we consider the matrix $\sum_{g \in A} (\mathbf{A}_M)_g \otimes \rho_i(g)$. Let $\rho = \rho_i$ and $f = f_i$. Then we have

$$\sum_{g \in A} (\mathbf{A}_M)_g \otimes \rho(g) = \begin{bmatrix} \sum_{g \in A} (\mathbf{A}_L)_g \otimes \rho(g) & \mathbf{B}_\rho \\ {}^{\mathrm{t}}\bar{\mathbf{B}}_\rho & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{B}_\rho {}^{\mathrm{t}}\bar{\mathbf{B}}_\rho - 2\mathbf{I}_{lf} & \mathbf{B}_\rho \\ {}^{\mathrm{t}}\bar{\mathbf{B}}_\rho & 0 \end{bmatrix}, \tag{5}$$

where

$$\mathbf{A}_M = \begin{bmatrix} \mathbf{A}_L & \mathbf{B} \\ {}^{\mathrm{t}}\mathbf{B} & 0 \end{bmatrix}.$$

We express the L-function of the middle graph $M(G)$ for a regular graph G in terms of characteristic polynomials.

Theorem 6. *Let G be a connected r-regular graph with n vertices and l edges, A a finite group and $\alpha : D(G) \to A$ an ordinary voltage assignment. Furthermore, let ρ be an irreducible representation of A, and f the degree of ρ. Suppose that the A-covering G^α of G is connected. Then*

$$\begin{aligned} &\mathbf{Z}_{M(G)}(u,\rho,\alpha_M)^{-1} \\ &= (1-u^2)^{(r-1)(l+n)f-lf} u^{nf} (1+u+(r-1)u^2)^{nf} (1+2u+(2r-1)u^2)^{(l-n)f} \\ &\quad \Phi\left(\sum_{g \in A} \rho(g) \otimes \mathbf{A}_g; \frac{1+(2-r)u+(2r-2)u^2+(r-1)(2-r)u^3+(2r-1)(r-1)u^4}{u(1+u+(r-1)u^2)} \right). \end{aligned}$$

Proof. For a vertex w of $M(G)$, we have

$$\deg_{M(G)} w = \begin{cases} r & \text{if } w \in V(G), \\ 2r & \text{if } w \in E(G). \end{cases}$$

Set $a=2r-1$ and $b=r-1$. By Theorem 1, we have

$$\mathbf{Z}_{M(G)}(u,\rho,\alpha_M)^{-1}$$
$$=(1-u^2)^{(r-1)(l+n)f-lf}\det\left(\mathbf{I}_{(l+n)f}-u\sum_{g\in A}\rho(g)\otimes(\mathbf{A}_M)_g+u^2(\mathbf{Q}_M)_f\right).$$

Then, by (5),

$$h(u):=\det\left(\mathbf{I}_{(l+n)f}-u\sum_{g\in A}\rho(g)\otimes(\mathbf{A}_M)_g+u^2(\mathbf{Q}_M)_f\right)$$
$$=\det\left(\mathbf{I}_{(l+n)f}-u\sum_{g\in A}(\mathbf{A}_M)_g\otimes\rho(g)+u^2\mathbf{Q}_M\otimes\mathbf{I}_f\right)$$
$$=\det\begin{bmatrix}(1+au^2)\mathbf{I}_{lf}-u\sum_{g\in A}(\mathbf{A}_L)_g\otimes\rho(g) & -u\mathbf{B}_\rho\\ -u{}^t\bar{\mathbf{B}}_\rho & (1+bu^2)\mathbf{I}_{nf}\end{bmatrix}.$$

By (2), we have

$$h(u)=\det\begin{bmatrix}(1+au^2)\mathbf{I}_{lf}-u(\mathbf{B}_\rho{}^t\bar{\mathbf{B}}_\rho-2\mathbf{I}_{lf}) & -u\mathbf{B}_\rho\\ -u{}^t\bar{\mathbf{B}}_\rho & (1+bu^2)\mathbf{I}_{nf}\end{bmatrix}$$
$$=\det\begin{bmatrix}(1+2u+au^2)\mathbf{I}_{lf}-\dfrac{u(1+u+bu^2)}{1+bu^2}\mathbf{B}_\rho{}^t\bar{\mathbf{B}}_\rho & *\\ 0 & (1+bu^2)\mathbf{I}_{nf}\end{bmatrix}$$
$$=(1+bu^2)^{(n-l)f}u^{lf}(1+u+bu^2)^{lf}$$
$$\det\left(\frac{(1+2u+au^2)(1+bu^2)}{u(1+u+bu^2)}\mathbf{I}_{lf}-\mathbf{B}_\rho{}^t\bar{\mathbf{B}}_\rho\right).$$

By (3) and (4),

$$h(u)=u^{nf}(1+u+bu^2)^{nf}(1+2u+au^2)^{(l-n)f}$$
$$\det\left(\frac{(1+2u+au^2)(1+bu^2)}{u(1+u+bu^2)}\mathbf{I}_{nf}-{}^t\bar{\mathbf{B}}_\rho\mathbf{B}_\rho\right)$$
$$=u^{nf}(1+u+bu^2)^{nf}(1+2u+au^2)^{(l-n)f}$$

$$\det\left(\frac{1+(2-r)u+(a+b-r)u^2+b(2-r)u^3+abu^4}{u(1+u+bu^2)}\mathbf{I}_{nf}\right.$$

$$\left.-\sum_{g\in A}\mathbf{A}_g\otimes\rho(g)\right)$$

$$=u^{nf}(1+u+bu^2)^{nf}(1+2u+au^2)^{(l-n)f}$$

$$\Phi\left(\sum_{g\in A}\mathbf{A}_g\otimes\rho(g);\frac{1+(2-r)u+(a+b-r)u^2+b(2-r)u^3+abu^4}{u(1+u+bu^2)}\right).$$

Therefore, the result follows. □

Mizuno and Sato [13] expressed the zeta function of the middle graph $M(G^\alpha)$ for a regular graph G by using the characteristic polynomial of G^α.

Corollary 2 (Mizuno and Sato [13]). *Let G be a connected r-regular graph with n vertices and l edges, A a finite group and $\alpha: D(G)\to A$ an ordinary voltage assignment. Set $|A|=m$. Suppose that the A-covering G^α of G is connected. Then*

$$\mathbf{Z}(M(G^\alpha),u)^{-1}$$

$$=(1-u^2)^{m(r-1)(l+n)-lm}u^{nm}(1+u+(r-1)u^2)^{nm}(1+2u+(2r-1)u^2)^{(l-n)m}$$

$$\Phi\left(G^\alpha;\frac{1+(2-r)u+(2r-2)u^2+(r-1)(2-r)u^3+(2r-1)(r-1)u^4}{u(1+u+(r-1)u^2)}\right).$$

Proof. Let $\rho_1=1,\ \rho_2,\dots,\rho_t$ be the irreducible representations of A, and f_i the degree of ρ_i for each i, where $f_1=1$. By Theorems 2, 5 and 6, it follows that

$$\mathbf{Z}(M(G^\alpha),u)^{-1}=\mathbf{Z}(M(G)^{\alpha_M},u)^{-1}=\prod_{i=1}^{t}\mathbf{Z}_{M(G)}(u,\rho_i,\alpha_M)^{-f_i}$$

$$=(1-u^2)^{m(r-1)(l+n)-lm}u^{nm}(1+u+bu^2)^{nm}(1+2u+au^2)^{(l-n)m}$$

$$\Phi(G;h_1(u))\prod_{i=2}^{t}\Phi\left(\sum_{g\in A}\rho_i(g)\otimes\mathbf{A}_g;h_1(u)\right)^{f_i},$$

where $a=2r-1,\ b=r-1$ and

$$h_1(u)=\frac{1+(2-r)u+(a+b-r)u^2+b(2-r)u^3+abu^4}{u(1+u+bu^2)}.$$

The result is obtained from Theorem 3. □

Now, we consider the previous example. Let $V(M(K_3))=\{[v_1],[v_2],[v_3],[e_{12}],[e_{23}],[e_{31}]\}$, where we set $[v_i]=v_iv_i'$, $i=1,2,3$. Then, the ordinary voltage assignment $\alpha_M : D(M(K_3))\to Z_3$ is given as follows:

$$\alpha_M(u,v)=\alpha_L(u,v),(u,v)\in D(L(K_3)); \qquad \alpha_M([v_2],[e_{12}])=\alpha(v_1,v_2)^{-1}=\tau^2,$$

$$\begin{aligned}\alpha_M([v_3],[e_{23}])&=\alpha_M([v_3],[e_{13}])=\alpha_M([v_1],[e_{12}])=\alpha_M([v_1],[e_{13}])\\ &= \alpha_M([v_2],[e_{23}])=1.\end{aligned}$$

It is certain that $M(K_3^{\alpha})=M(C_9)=M(K_3)^{\alpha_M}$. By Theorem 6, we have

$$\mathbf{Z}_{M(K_3)}(u,\chi_2,\alpha_M)^{-1}=(1-u^2)^3u^3(1+u+u^2)^3\Phi\left(\sum_{i=0}^{2}\chi_2(\tau^i)\mathbf{A}_{\tau^i};\frac{1+2u^2+3u^4}{u+u^2+u^3}\right).$$

Since $\Phi(\sum_{i=0}^{2}\chi_2(\tau^i)\mathbf{A}_{\tau^i};\lambda)=\lambda^3-3\lambda+1$, it follows that

$$\begin{aligned}\mathbf{Z}_{M(K_3)}(u,\chi_2,\alpha_M)^{-1}&=(1-u^2)^3\\ &\quad(1-u+3u^2-8u^3+10u^4-20u^5+21u^6\\ &\quad -36u^7+22u^8-32u^9+39u^{10}-9u^{11}+27u^{12}).\end{aligned}$$

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